

# ON HANDLEBODY STRUCTURES OF RATIONAL BALLS

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**ABSTRACT.** It is known that for coprime integers  $p > q \geq 1$ , the lens space  $L(p^2, pq-1)$  bounds a rational ball,  $B_{p,q}$ , arising as the 2-fold branched cover of a (smooth) slice disk in  $B^4$  bounding the associated 2-bridge knot. Lekilli and Maydanskiy [LM12] give handle decompositions for each  $B_{p,q}$ . Whereas, Yamada [Yam07] gives an alternative definition of rational balls,  $A_{m,n}$ , bounding  $L(p^2, pq-1)$  by their handlebody decompositions alone. We show that these two families coincide - answering a question of Kadokami and Yamada in [KY14]. To that end, we show that each  $A_{m,n}$  admits a Stein filling of the “standard” contact structure,  $\bar{\xi}_{st}$ , on  $L(p^2, pq-1)$  investigated by Lisca in [Lis08].

## 1. INTRODUCTION

For  $p > q \geq 1$  relatively prime, let  $B_{p,q}$  be the 4-manifold obtained by attaching a 1-handle and a single 2-handle with framing  $pq-1$  to  $B^4$  by wrapping the attaching circle of the 2-handle  $p$ -times around the 1-handle with a  $q/p$ -twist (see the left side of Figure 1). From this description, it is immediate that  $B_{p,q}$  is always a rational homology ball. Lekili and Maydanskiy [LM12] show that each such  $B_{p,q}$  arises as the 2-fold branched cover of  $B^4$  branched over a slice disk for the (slice) 2-bridge knot associated to the fraction  $-p^2/(pq-1)$ . That is, the family  $B_{p,q}$  represents handle decompositions of the rational balls introduced by Casson and Harer in [CH81]. As such,  $\partial B_{p,q} \approx L(p^2, pq-1)$  - throughout  $\approx$  denotes when two manifolds are diffeomorphic.

In a similar direction, Yamada [Yam07] defines a family of rational balls bounding  $L(p^2, pq-1)$  directly via their handle decompositions: For  $n, m \geq 1$  relatively prime, let  $A_{m,n}$  be the 4-manifold obtained by attaching a 1-handle and a single 2-handle with framing  $mn$  to  $B^4$  by attaching the 2-handle along a simple closed curve embedded on a once-punctured torus viewed in  $S^1 \times S^2$  so that the attaching circle traverses the two 1-handles of the torus  $m$  and  $n$  times respectively (see the right side of Figure 1). Yamada goes on to define an involutive symmetric function,  $A$ , on the set of coprime pairs of positive integers such that if  $A(p-q, q) = (m, n)$  then  $\partial A_{m,n} \approx L(p^2, pq-1)$  (see Lemma 5.1 for a definition of  $A$ ).

Given these two constructions of rational balls with coincident boundaries, one arrives at a natural question posed by Kadokami and Yamada in [KY14] as Problem 1.9: When are these two families diffeomorphic, homeomorphic, or even homotopic relative to their boundaries as 4-manifolds? We provide a complete answer to this question by proving the following theorem.

**Theorem 1.1.** *For each pair of relatively prime positive integers,  $(m, n)$ ,  $A_{m,n}$  carries a Stein structure,  $\tilde{J}_{m,n}$ , filling a contact structure contactomorphic to the standard contact structure  $\bar{\xi}_{st}$  on the lens space  $\partial A_{m,n}$ . In particular, each  $A_{m,n} \approx B_{p,q}$  if and only if  $\partial A_{m,n} \approx \partial B_{p,q}$ .*

The proof of Theorem 1.1 follows by first explicitly writing down a Stein structure on  $A_{m,n}$  using Eliashberg and Gompf’s [Gom98] characterization of handle decompositions of Stein domains. Then, verifying that the homotopy invariants of the induced contact structures on the boundary agree with those of  $(L(p^2, pq-1), \bar{\xi}_{st})$ , showing that the two structures are homotopic as 2-plane fields. Work of Honda’s [Hon00] shows that this is sufficient to conclude that these two contact structures are contactomorphic. Lisca’s classification [Lis08] of the diffeomorphism types of symplectic fillings of  $(L(p^2, pq-1), \bar{\xi}_{st})$  then gives that  $A_{m,n} \approx B_{p,q}$ . In order to successfully

compare the aforementioned homotopy invariants, we construct boundary diffeomorphisms. These boundary diffeomorphisms can be extended to explicit diffeomorphisms between  $B_{p,q}$  and  $A_{m,n}$  through the carving process introduced in [Akb77]; in fact, we have:

**Theorem 1.2.** *Let  $(m, n) = A(p - q, q)$  for some  $p > q > 0$  relatively prime. Then there exists a diffeomorphism  $f : \partial B_{p,q} \rightarrow \partial A_{m,n}$  such that  $f$  carries the belt sphere,  $\mu_1$ , of the single 2-handle in  $B_{p,q}$  to an unknot in  $\partial A_{m,n}$  (see Figure 1). Moreover, carving  $A_{m,n}$  along  $f(\mu_1)$  gives  $S^1 \times B^3$ .*

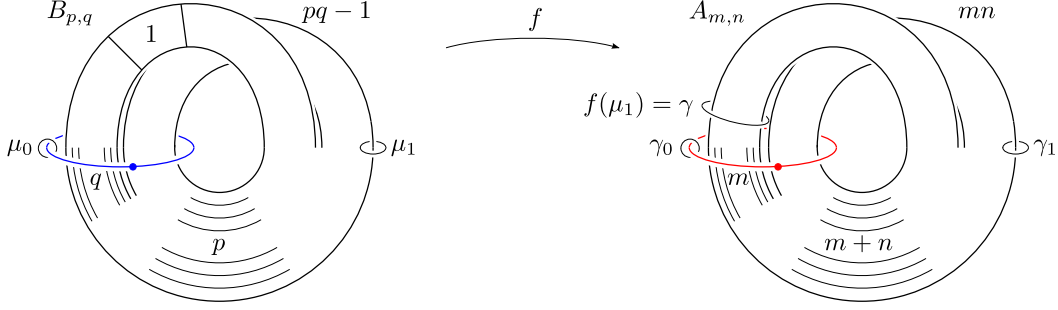


FIGURE 1. The spaces  $B_{p,q}$  and  $A_{m,n}$ .

**Corollary 1.3.**  *$f$  extends to a diffeomorphism  $\tilde{f} : B_{p,q} \rightarrow A_{m,n}$ .*

In [FS97], Fintushel and Stern define a smooth operation, the rational blow-down, on 4-manifolds containing certain configurations of spheres by removing a neighborhood of those spheres and replacing them by the rational ball  $B_{p,p-1}$ . In [Par97], Park generalized the operation to a larger set of configurations at the expense of having to glue in  $B_{p,q}$  for  $q$  other than  $p - 1$ . In the presence of a symplectic structure (and a symplectic configuration of spheres), both operations can be performed symplectically [Sym98, Sym01]. Moreover, under mild assumptions (see [FS97], [Par97] for details), nontrivial solutions to the Seiberg-Witten equations on the original 4-manifold induce nontrivial solutions on the surgered manifold.

Therefore, having well understood handle decompositions for  $B_{p,q}$  allows one to construct explicit examples of rationally blown-down 4-manifolds. For instance, this has been used to construct an exotic  $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$  in [SS05]. Corollary 1.3 and Theorem 1.1 are then useful, since either the decomposition  $B_{p,q}$  or  $A_{m,n}$  can conceivably be used interchangeably.

**1.1. Conventions and Assumptions.** Unless specifically stated to the contrary, throughout the paper, we assume  $p - q > q \geq 1$ ,  $n > m \geq 1$ , and that both pairs are relatively prime. As  $B_{p,q} \approx B_{p,p-q}$  and  $A_{m,n} \approx A_{n,m}$ , this assumption doesn't represent a restriction. We adopt the standard convention that  $L(p, q)$  is the result of  $-p/q$ -surgery on the unknot in  $S^3$ . It is well known that  $L(p, q)$  is also given as the boundary of a linear plumbing of  $D^2$ -bundles over  $S^2$  with Euler classes chosen according to the continued fraction associated to  $-p/q$ :

$$[c_1, \dots, c_n] \doteq c_1 - \frac{1}{c_2 - \frac{1}{\ddots - \frac{1}{c_n}}} = -\frac{p}{q}$$

where,  $c_i$  are uniquely determined provided  $c_i \leq -2$ . We will often forgo the uniqueness of the  $c_i$ 's in favor of shorter continued fraction expansions and thus smaller bounding 4-manifolds.

The continued fraction associated to  $-p^2/(pq - 1)$  involves the Euclidean algorithm (see [CH81] as well as [Yam07]). Therefore, we use the Euclidean algorithm to define sequences of remainders and divisors of  $p$  and  $q$  as follows:

**Definition 1.4.** For  $p > q \geq 1$ , relatively prime, let  $\{r_i\}_{i=-1}^{\ell+2}$  and  $\{s_i\}_{i=0}^{\ell+1}$  be defined recursively by  $r_{-1} \doteq p$ ,  $r_0 \doteq q$  and

$$r_{i+1} = r_{i-1} \mod r_i, \quad r_{i-1} = r_i s_i + r_{i+1}.$$

Let  $\ell$  be the last index where  $r_\ell > 1$  so that  $r_{\ell+1} = 1$  and  $r_{\ell+2} \doteq 0$ .

For bookkeeping purposes, we'll differentiate between the above sequences for  $p$  and  $q$  and the analogously defined sequences  $\{\rho_i\}_{i=-1}^{\ell+2}$  and  $\{\sigma_i\}_{i=0}^{\ell+1}$  associated to  $n > m \geq 1$ . Furthermore, provided that  $p - q > q$ ,  $\ell$  agrees between the two sequences when  $A(p - q, q) = (m, n)$  or  $(n, m)$  (see Remark 3.8 and Lemma 5.1).

**1.2. Organization.** The paper is organized as follows: In Section 2, we construct Stein structures on each  $A_{m,n}$  using Eliashberg and Gompf's characterization of handle decompositions of Stein domains. In Section 3, we construct explicit boundary diffeomorphisms from  $B_{p,q}$  and  $A_{m,n}$  to their lens space boundaries - proving Theorem 1.2. In Section 4, we prove Theorem 1.1 by using those boundary diffeomorphisms to determine which contact structures are induced by the Stein structures of Section 2. For clarity we relegate much of the required algebra to Section 5.

## 2. STEIN STRUCTURES ON $A_{m,n}$

In this section, we show that  $A_{m,n}$  admits a Stein structure. To accomplish this, we use the handle characterization of Stein surfaces given in [Gom98]. The reader can also consult [GS99] as well as [OS04] for thoughtful treatments of the subject. Such a Stein structure induces a (tight) contact structure on  $\partial A_{m,n}$ . Tight contact structures on lens spaces are well understood; Honda completely classifies them in [Hon00]. Moreover, in [Lis08], Lisca classifies the diffeomorphism types of symplectic fillings of  $(L(p, q), \bar{\xi}_{st})$  where  $\bar{\xi}_{st}$  is the contact structure  $L(p, q)$  inherits from the unique tight contact structure on  $S^3$  via the cyclic group action. In particular, Lisca defines 4-manifolds  $W_{p,q}(\mathbf{n})$ , such that

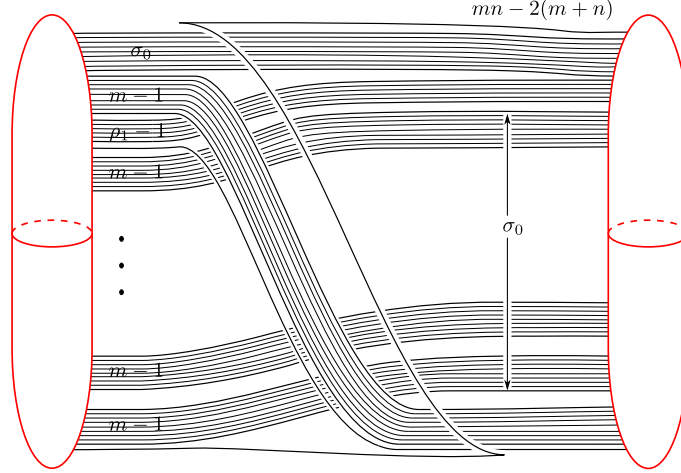
**Theorem 2.1** ([Lis08], Theorem 1.1). *Let  $p > q \geq 1$  be relatively prime integers. Then each symplectic filling  $(W, \omega)$  of  $(L(p, q), \bar{\xi}_{st})$  is orientation preserving diffeomorphic to a smooth blowup of  $W_{p,q}(\mathbf{n})$  for some  $\mathbf{n} \in \mathbf{Z}_{p,q}$ . Moreover, if  $b_2(W) = 0$ , then  $W$  is unique.*

In light of Lisca's theorem, if we show that not only does  $A_{m,n}$  admit a Stein structure, but that such a structure gives a symplectic filling of  $(L(p^2, pq - 1), \bar{\xi}_{st})$ , then we immediately have that  $A_{m,n} \approx B_{p,q}$  since it is known that  $B_{p,q}$  admits a Stein structure giving such a filling. Indeed, by sliding the 2-handle of  $B_{p,q}$  over the 1-handle  $q$ -times one arrives at the Stein domain,  $(B_{p,q}, J_{p,q})$ , given in Figure 22 and investigated in [LM12]; there, the authors prove that  $J_{p,q}$  fills the standard contact structure on  $L(p^2, pq - 1)$ .

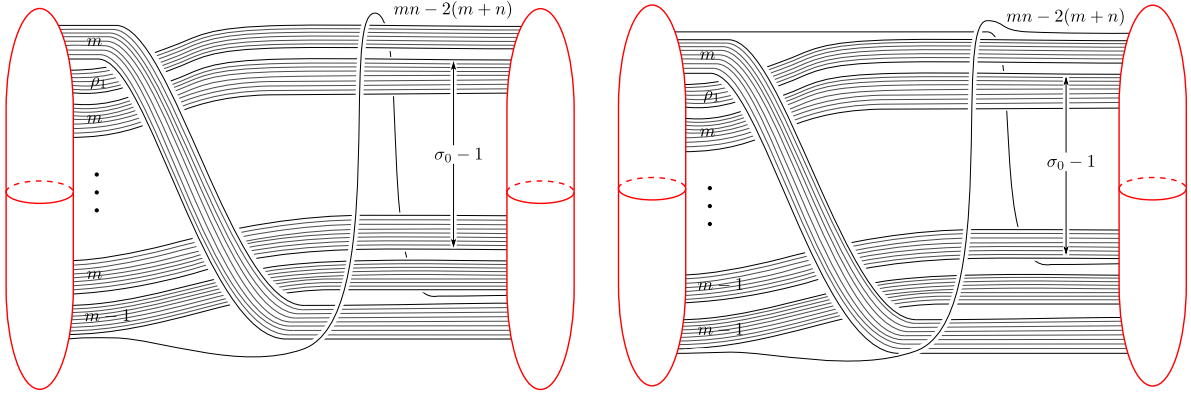
**Proposition 2.2.** *Each  $A_{m,n}$  admits a Stein structure,  $\tilde{J}_{m,n}$ , specified by Figure 2 where we assume  $n = m\sigma_0 + \rho_1$ .*

*Proof.* Notice that there are  $(m - 1)((\rho_1 - 1) + \sigma_0(m - 1)) + \sigma_0(m - 1)$  positive crossings,  $m + n - 1$  negative crossings and one left cusp coming from the Legendrian attaching circle,  $K \subset S^1 \times S^2$ , of the 2-handle in Figure 2. Then, the Thurston-Bennequin framing of  $K$  is

$$\begin{aligned} \text{tb}(K) &= \# \text{ of positive crossings} - \# \text{ of negative crossings} - \# \text{ of left cusps} \\ &= (m - 1)(\rho_1 - 1) + m\sigma_0(m - 1) - (m + n - 1) - 1 \\ &= m(m\sigma_0 + \rho_1) - m - m\sigma_0 - \rho_1 - (m + n) + 1 \\ &= mn - 2(m + n) + 1. \end{aligned}$$

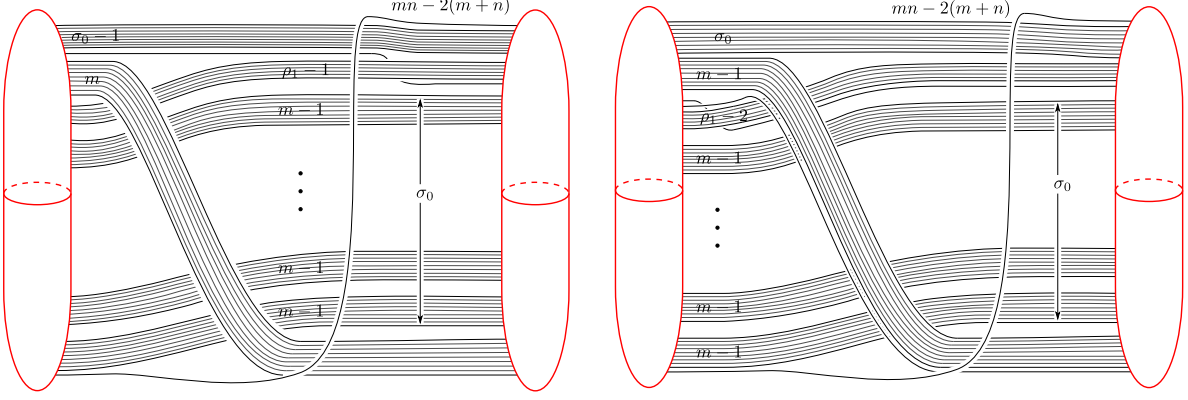
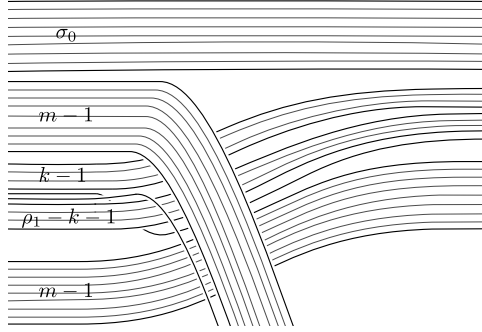
FIGURE 2.  $(A_{m,n}, \tilde{J}_{m,n})$ 

Then, [Gom98] gives that the unique Stein structure on  $S^1 \times B^3$  extends to  $A_{m,n}$  - provided that Figure 2 specifies  $A_{m,n}$ . To that end, express  $A_{m,n}$  in 2-ball notation and slide the 2-handle once under the 1-handle (left side of Figure 3). We refer to the portion of  $K$  passing behind the

FIGURE 3. Sliding the 2-handle of  $A_{m,n}$  under the 1-handle; dragging the attaching circle of that 2-handle around the 1-handle once.

central plane of the two attaching balls of the 1-handle as the “bad” strand. We now pair off negative crossings in the bad strand with positive crossings in  $K$  by “unraveling” the 2-handle. To accomplish this, begin by dragging the bad strand over the 1-handle (right side of Figure 3). Each time we drag the bad strand over the 1-handle, we unwind a strand off of the lowest remaining band of  $m$  strands and wind that strand into a parallel band at the top - thereby eliminating  $m - 1$  negative crossings at the expense of  $m - 1$  positive crossings. Repeating  $\sigma_0 - 1$  more times gives the left side of Figure 4. We again push what remains of the bad strand around the 1-handle - this time, a total of  $\sigma_0 + 2$ -times - giving the right side of Figure 4. We repeat the process of dragging the negative twist around the 1-handle  $\sigma_0 + 2$  times. Each time, the twist involves one less strand. After  $k$  such iterations, the braid in the upper right of Figure 4 is replaced by that of Figure 5. It’s then immediate that  $\rho_1 - 1$  iterations gives Figure 2.  $\square$

The fact that  $(\partial A_{m,n}, \xi_{J_1})$  is contactomorphic to  $(\partial B_{p,q}, \xi_{J_{p,q}})$  and thus to  $(L(p^2, pq - 1), \bar{\xi}_{\text{st}})$  is Corollary 4.7. Also, it is worth noting that  $\tilde{J}_{1,p-1}$  is  $J_{p-1,1}$ .

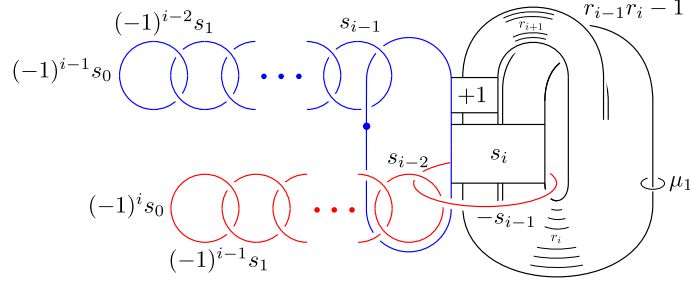
FIGURE 4. The result of dragging the attaching circle  $\sigma_0$ -times; and after  $2\sigma_0 + 2$ -times.FIGURE 5. The result of dragging the 2-handle of  $A_{m,n}$  around the 1-handle  $(k + 1)\sigma_0 + 2$ -times.

### 3. BOUNDARY DIFFEOMORPHISMS

In this section, we exhibit explicit diffeomorphisms from  $\partial B_{p,q}$  and  $\partial A_{m,n}$  to  $L(p^2, pq - 1)$ . To accomplish this, we find boundary diffeomorphisms to particular linear plumbings associated to  $p$  and  $q$  (respectively  $m$  and  $n$ ). These diffeomorphisms are needed to compare the resulting homotopy invariants of the contact structures induced by the Stein structures on  $B_{p,q}$  of [LM12] as well as those on  $A_{m,n}$  coming from Proposition 2.2. Along the way, we trace the meridian of the attaching circle of the single 2-handle of  $B_{p,q}$  - proving Theorem 1.2.

It's worth noting that such diffeomorphisms have been known previously. In [Yam07], Yamada produces diffeomorphisms from  $\partial A_{m,n}$  to  $L(p^2, pq - 1)$  expressed as the boundary of the unique linear plumbing of  $D^2$ -bundles over  $S^2$  with Euler classes each  $\leq -2$ . To accomplish this, one must carefully keep track of every stage of the Euclidean algorithm applied to  $(p - q, q) = 1$  - that is every time  $a_i$  is subtracted from  $b_i$  or  $b_i$  from  $a_i$  in Yamada's definition of  $A(p - q, q)$  (see Lemma 5.1). We perform a courser bookkeeping of the Euclidean algorithm via Definition 1.4, which allows for an arguably clearer definition - however, we don't arrive at a linear plumbing with Euler classes  $\leq -2$ . Yet, through a sequence of blow-ups and cancellations, one can easily get to that plumbing if so desired. Furthermore, this definition lends itself to defining the diffeomorphism from  $\partial B_{p,q}$  to  $L(p^2, pq - 1)$  as well:

**Proposition 3.1.** *Let  $\{r_i\}_{i=-1}^{\ell+2}$  and  $\{s_i\}_{i=0}^{\ell+1}$  be as defined in Definition 1.4. Then for each  $i \in \{0, \dots, \ell + 1\}$ ,  $B_{p,q} \approx^{\partial} B_{p,q}^i$  where  $B_{p,q}^i$  is the 4-manifold given by Figure 6.*

FIGURE 6. The 4-manifold  $B_{p,q}^i$ 

*Proof.* We induct on  $i$ . When  $i = 0$ , the result is immediate since  $B_{p,q}^0 \approx B_{p,q}$ . Therefore, the proposition holds provided that  $\partial B_{p,q}^i \approx \partial B_{p,q}^{i+1}$ . Let  $K_1^i$  be the attaching circle of the  $r_{i-1}r_i - 1$ -framed 2-handle in  $B_{p,q}^i$ . Suppose the result holds for some  $i \leq \ell$ . For  $i + 1$ , first, surger the single 1-handle and introduce a canceling pair of 1- and 2-handles to remove the  $s_i$ -full twists between  $K_1^i$  and the, now surgered, 1-handle (Figure 7). Since  $K_1^i$  links the new 1-handle  $r_i$

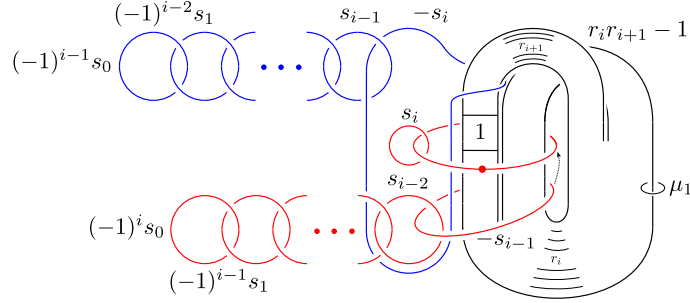
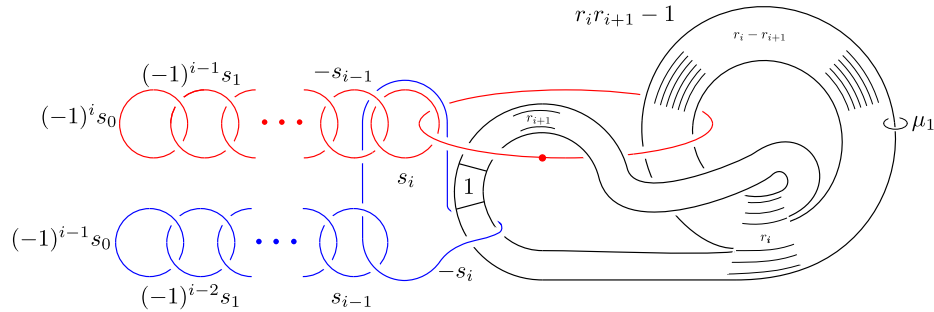


FIGURE 7. Introducing a canceling pair after surgery.

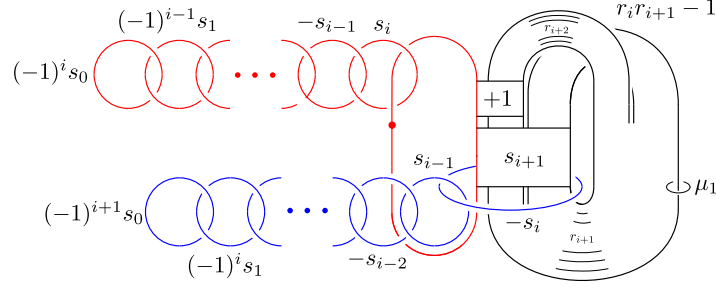
times, the framing on  $K_1^i$  decreases by  $s_i r_i^2$  and the new framing on  $K_1^i$  is

$$r_{i-1}r_i - 1 - s_i r_i^2 = r_i(r_{i-1} - s_i r_i) - 1 = r_i r_{i+1} - 1.$$

Sliding the  $-s_{i-1}$ -framed 2-handle under the new 1-handle as indicated in Figure 7, and isotoping the  $r_{i+1}$ -stranded band (see Figure 8) we find that the  $r_{i+1}$ -stranded band traverses the

FIGURE 8. Isotoping  $K_1^i$ .

1-handle (positively)  $s_{i+1}$ -times as a complete band, while  $r_{i+2}$ -strands traverse an additional one time to make up the complete  $s_{i+1}r_{i+1} + r_{i+2} = r_i$  linking. With this view in mind, we isotope  $K_1^i$  into a closed braid on  $r_{i+1}$  strands appropriately linking the carving disk of the 1-handle - Figure 9. The result holds by induction.  $\square$

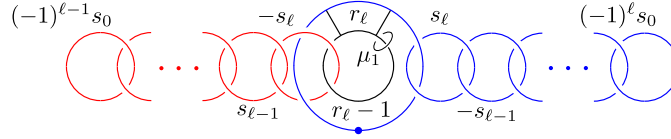
FIGURE 9. Further isotopy of  $K_1^i$  to  $K_1^{i+1}$ 

**Remark 3.2.** At no point does  $\mu_1$ , the meridian of  $K_1^i$ , get damaged under the boundary diffeomorphisms defined in Proposition 3.1. In particular, for each  $i$ ,  $\mu_1$  bounds a disk in  $B_{p,q}^i$  and the image of a collar neighborhood of  $\mu_1$  arising from such a disk persists under the boundary diffeomorphisms defined above - that is that each diffeomorphism preserves the 0-framing on  $\mu_1$ .

Since  $r_{\ell+1} = 1$  and  $r_{\ell+2} = 0$ , by definition,  $s_{\ell+1} = s_{\ell+1}r_{\ell+1} + r_{\ell+2} = r_{\ell}$ . So, by looking at  $B_{p,q}^{\ell+1}$  we arrive at the following result of Casson and Harer [CH81].

**Corollary 3.3.**  $\partial B_{p,q} \approx L(p^2, pq - 1)$ .

*Proof.* By Proposition 3.1, we have that  $\partial B_{p,q} \approx \partial B_{p,q}^{\ell+1}$  (Figure 10). We show that  $\partial B_{p,q}^{\ell+1}$  is

FIGURE 10. The space  $B_{p,q}^{\ell+1}$ .

diffeomorphic to a linear plumbing of disk-bundles over  $S^2$  as follows. Surger the 1-handle and introduce a canceling 1- and 2-handle, as in the induction step of Proposition 3.1, (top of Figure 11). Next, slide the  $-s_{\ell}$ -framed 2-handle as well as  $\mu_1$  under the 1-handle as indicated in the top of Figure 11 (middle of Figure 11). Surgering the new 1-handle and blowing down gives the linear plumbing (bottom of Figure 11).  $\square$

**Remark 3.4.** From Lemma 5.4, we see that the above linear plumbing bounds  $L(p^2, pq - 1)$ . Indeed

$$[-s_0, s_1, \dots, \pm r_{\ell}, 1, \mp r_{\ell}, \dots, -s_1, s_0] = -\frac{p^2}{pq - 1}.$$

Notice also that the image of  $\mu_1$  is given as the 0-framed push-off of the attaching circle of the central 1-framed unknot. We'll trace where the curve,  $\gamma$  in Figure 1, goes as well - finding that it too goes to the 0-framed push-off of the central 1-framed unknot via an appropriately defined diffeomorphism. To define this diffeomorphism, in a structurally similar manner to that of Proposition 3.1, we note the following fact about  $A_{m,n}$ .

**Lemma 3.5.**  $A_{m,n}$  is given by Figure 12.

*Proof.* As before, we are taking  $n = m\sigma_0 + \rho_1$ . The result follows from an isotopy of the 2-handle.  $\square$

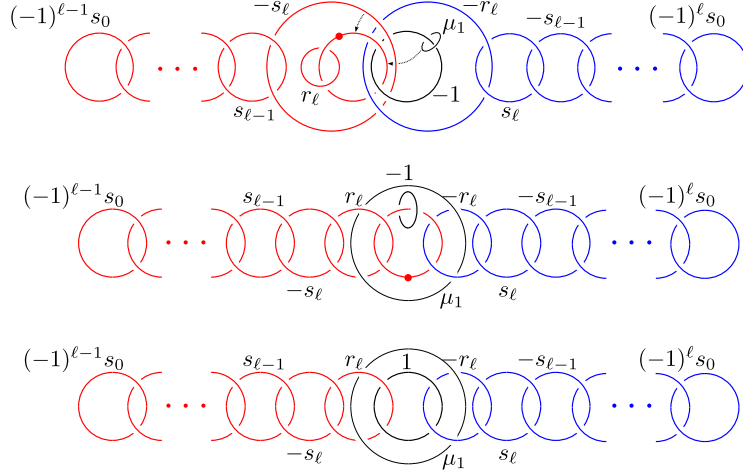


FIGURE 11. From top to bottom: The introduction of a canceling pair to  $B_{p,q}^{\ell+1}$  after surgery; the result of the indicated slides; a linear plumbing associated to  $\partial B_{p,q}$ .

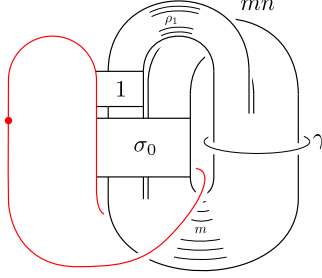


FIGURE 12. An alternative description of  $A_{m,n}$ .

**Proposition 3.6.** Let  $\{\rho_i\}_{i=-1}^{\ell+2}$  and  $\{\sigma_i\}_{i=0}^{\ell+1}$  be as defined in Definition 1.4 (associated to  $n > m \geq 1$ ). Then for each  $i \in \{0, \dots, \ell+1\}$ ,  $A_{m,n} \stackrel{\partial}{\approx} A_{m,n}^i$  where  $A_{m,n}^i$  is the 4-manifold given by Figure 13.

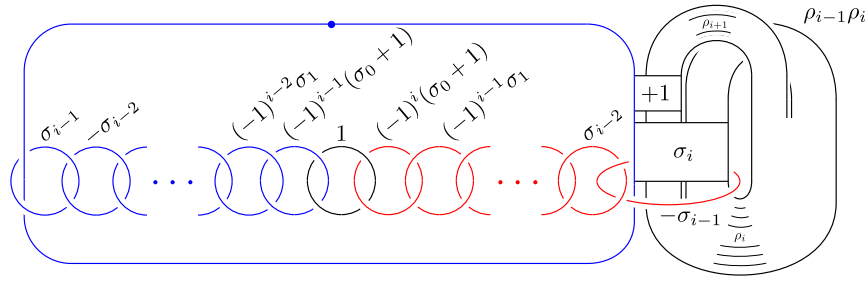


FIGURE 13. The 4-manifold  $A_{m,n}^i$

*Proof.* We induct on  $i$ , treating the base case and the induction step simultaneously. For the base case, start with the handle decomposition from Lemma 3.5. For the induction step, suppose that the result holds for some  $i \leq \ell$ . Let  $K_1^i$  be the attaching circle of the  $\rho_{i-1}\rho_i$ -framed 2-handle in  $A_{m,n}^i$ . Surger the 1-handle and introduce a canceling 1- and 2-handle (for the base case see the left side of Figure 14, for the induction step see Figure 15). Notice, similar to



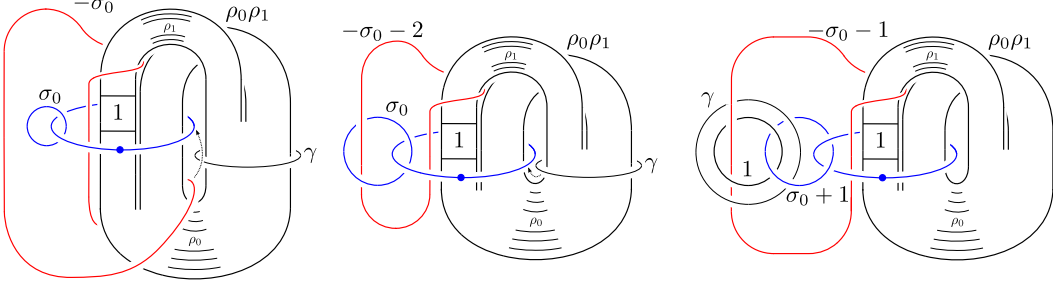


FIGURE 14. The base case of Proposition 3.6

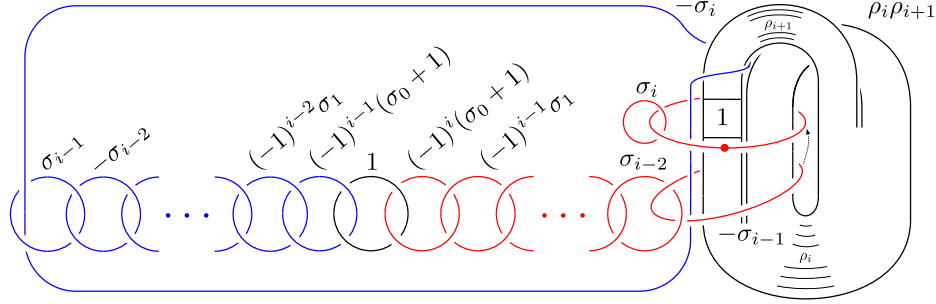
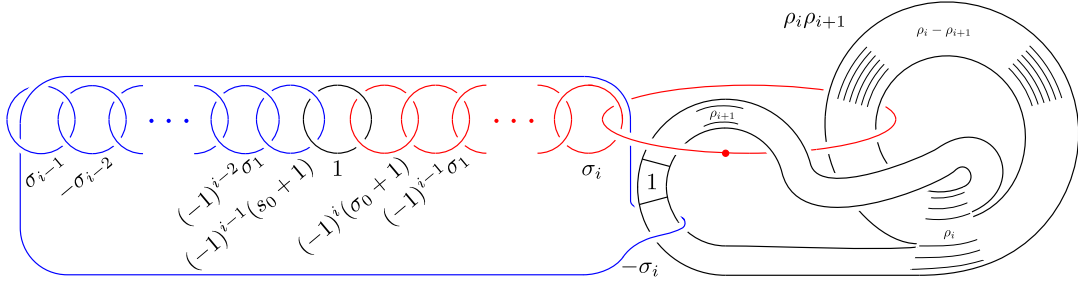


FIGURE 15. Introducing a canceling pair.

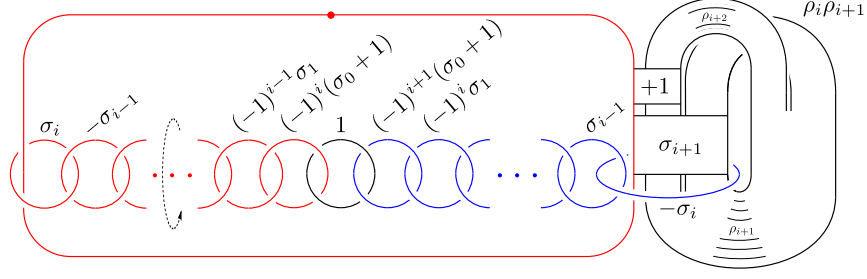
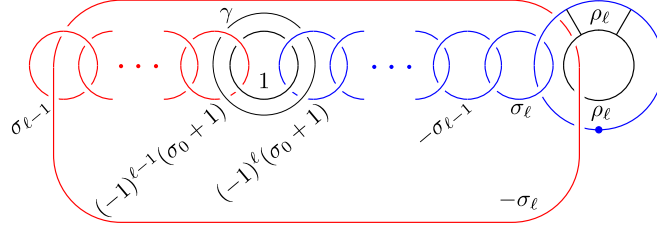
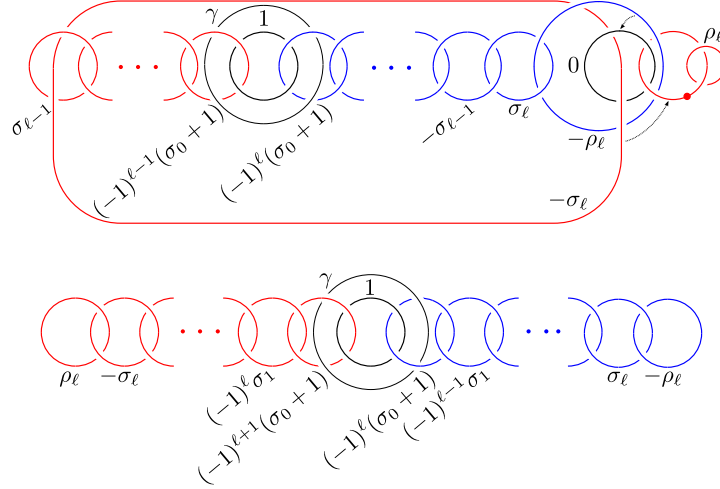
Proposition 3.1 the framing of  $K_1^i$  changes from  $\rho_{i-1}\rho_i$  to  $\rho_i\rho_{i+1}$ . Slide the now surgered 1-handle as indicated in the respective figures and, for the base case, blow-up once (right side of Figure 14). From here the base case follows similarly to the induction step; both of which are structurally similar to Proposition 3.1. Indeed, isotope  $K_1^i$  to view a band with  $\rho_{i+1}$  strands traversing the 1-handle  $\sigma_{i+1}$ -times along with  $\rho_{i+2}$  of those strands traversing an extra time as in Figure 16. A further isotopy of  $K_1^i$  gives a closed braid on  $\rho_{i+1}$ -strands geometrically linking

FIGURE 16. Isotoping  $K_1^i$  in  $A_{m,n}^i$ .

the carving disk of the new 1-handle  $\rho_i$ -times. Finally, notice that to get the appropriate linking on the chain of unknots, we have to wind the chain (as indicated in Figure 17) to add a total of  $i$  positive half-twists to the left of the euler-class 1 disk-bundle along with  $i$  negative half-twists to the right. The result follows by induction.  $\square$

**Corollary 3.7** ([Yam07] Theroem 1.1).  $\partial A_{m,n} \approx L(p^2, pq - 1)$  for  $(p - q, q) = A(m, n)$ .

*Proof.* By Proposition 3.6,  $\partial A_{m,n} \approx \partial A_{m,n}^{\ell+1}$  (figure 18). We proceed as in Corollary 3.3. After surgering the 1-handle and introducing a canceling 1- and 2-handle (top of Figure 19), slide

FIGURE 17. Further isotopy of  $K_1^i$  to  $K_1^{i+1}$  in  $A_{m,n}^{i+1}$ .FIGURE 18. The space  $A_{m,n}^{\ell+1}$ FIGURE 19. The result of surgering  $A_{m,n}^{\ell+1}$  and introducing a canceling pair; a linear plumbing associated to  $\partial A_{m,n}$ 

the  $-\sigma_\ell$ -framed 2-handle under the 1-handle and the  $-\rho_\ell$ -framed 2-handle over the 0-framed 2-handle as indicated in the top of Figure 19. Canceling the 1-handle with the 0-framed 2-handle gives the linear plumbing (bottom of Figure 19).  $\square$

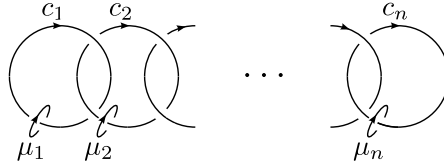
**Remark 3.8.** The fact that  $\partial A_{m,n}$  is  $L(p^2, pq - 1)$  for  $A(m, n) = (p - q, q)$  follows by noting that given  $p$  and  $q$ , or equivalently  $m$  and  $n$ , we can define the other pair by an appropriate identification of the linear plumbings in Corollaries 3.3 and 3.7 - provided that  $s_0 > 1$  (that is that  $p - q > q$ ). In fact, this could be taken as the definition of the function  $A$  defined in [Yam07]. The latter claim is the content of Lemma 5.1. Notice also that  $\gamma$  bounds a disk in each  $\partial A_{m,n}^i$  as well as in the linear plumbing of Figure 19. Furthermore, each boundary

diffeomorphism defined in Proposition 3.6 and those of Corollary 3.7 preserve the 0-framing of  $\gamma$  specified by those disks.

*Proof of Theorem 1.2.* As  $A(p-q, q) = (m, n)$ , we can identify the plumbings of Figures 11 and 19. Then, by first, applying the diffeomorphisms of Proposition 3.1 we get a diffeomorphism from  $\partial B_{p,q}$  to the boundary of the linear plumbing of the bottom of Figure 11 caring  $\mu_1$  as indicated. Then applying the diffeomorphisms of Proposition 3.6 in reverse from the boundary of the linear plumbing of Figure 19 to  $A_{m,n}$  gives the required diffeomorphism  $f : \partial B_{p,q} \rightarrow \partial A_{m,n}$ . The fact that carving the disk bounding  $f(\mu_1)$  gives  $S^1 \times B^3$  follows by repeatedly sliding the now two 1-handles past each other and canceling one with the single 2-handle of  $A_{m,n}$ .  $\square$

**3.1. Spin Structures and Orientations.** We determine how  $f$  behaves with respect to elements of  $H_1(\partial B_{p,q})$  as well as how  $f$  treats spin structures. Both of these behaviors will be important.

**Lemma 3.9.** *Suppose that  $L(p, q)$  is given by the linear plumbing*



where the  $\mu_i$ 's are meridians spanning  $H_1(L(p, q), \mathbb{Z})$ . Then

$$H_1(L(p, q), \mathbb{Z}) = \langle \mu_1 : (\det C_n) \mu_1 = 0 \rangle$$

$$\text{where } C_i \doteq \begin{pmatrix} c_1 & 1 & & \\ 1 & c_2 & 1 & \\ & 1 & \ddots & 1 \\ & & 1 & c_i \end{pmatrix} \text{ and for } i \in \{2, \dots, n\}, \mu_i = (-1)^{i-1} (\det C_{i-1}) \mu_1.$$

*Proof.* Given a Dehn surgery description of a 3-manifold, one obtains a presentation for the first homology in terms of the right handed meridians of the (oriented) framed link (see [GS99] page 165). In the above case, we find that

$$H_1(L(p, q), \mathbb{Z}) = \langle \mu_1, \dots, \mu_n : \mu_2 = -c_1 \mu_1, \{\mu_{i+1} = -c_i \mu_i - \mu_{i-1}\}_{i=2}^{n-1}, c_n \mu_n = -\mu_{n-1} \rangle$$

As  $\mu_2 = -c_1 \mu_1 = (-1)^{2-1} (\det C_{2-1}) \mu_1$ , the result follows by induction using that

$$\det C_k = c_k \det C_{k-1} - \det C_{k-2}.$$

$\square$

**Remark 3.10.** Lemma 3.9 allows us to determine  $f_*^{-1} \gamma_0 \in H_1(\partial B_{p,q})$ . From Proposition 3.6, we have that a meridian of  $-(\sigma_0 + 1)$ -framed unknot of figure 19 is carried to  $\gamma_0$  in  $\partial A_{m,n}$ . Similarly,  $\mu_0$  is carried to a meridian of  $-s_0$ -framed unknot of Figure 11. Furthermore, by Corollary 5.6, we have that  $\gamma_0 = \pm n \mu_0$  if  $\ell \in 2\mathbb{Z}$  and  $\gamma_0 = \pm m \mu_0$  if  $\ell \in 2\mathbb{Z} + 1$  where we view  $\gamma_0$  and  $\mu_0$  as their respective images in the aforementioned linear plumbings. Now, by an appropriate choice of identification of the plumbings of Figures 19 and 11 we can always assume that

$$f_*^{-1} \gamma_0 = \begin{cases} +n \mu_0 & \text{if } \ell \in 2\mathbb{Z}, \\ +m \mu_0 & \text{if } \ell \in 2\mathbb{Z} + 1. \end{cases}$$

Indeed, if as defined,  $f_*^{-1} \gamma_0$  was  $-m \mu_0$  or  $-n \mu_0$ , we can simply flip one plumbing over before making the identification and redefine  $f$  accordingly!

Recall that  $L(p^2, pq - 1)$  admits a unique spin structure if  $p$  is odd and two spin structures if  $p$  is even. In the former case,  $f$  clearly maps the unique spin structure to itself. In the later case, we investigate how  $f$  behaves on spin structures by looking at characteristic sublinks:

**Definition 3.11** ([Kap79], Definition 1.10). For a framed link  $L \subset S^3$ , a sublink  $L' \subset L$  is characteristic if for each  $K \subset L$ ,

$$\ell k(K, L') = \ell k(K, K) \pmod{2}.$$

When  $M^3$  is given as (integral) surgery on  $L$ , spin structures on  $M$  are in bijection with characteristic sublinks of  $L$ . Furthermore, fixing a spin structure and thus a characteristic sublink of  $M$ , one can trace where that structure goes under a diffeomorphism specified via handle moves / blow-ups by tracing how the sublink evolves under those moves (see §5.7 of [GS99]). To accomplish this, we adopt the following notation to specify  $(M, \mathfrak{s})$  for  $\mathfrak{s} \in \mathcal{S}(M)$  - the set of spin structures on  $M$ :

**Notation 3.12.** If  $M^3$  is given by integral surgery on a framed link  $L = K_1^{f_1} \cup \dots \cup K_N^{f_N}$  with framings  $f_i \in \mathbb{Z}$  and  $\mathfrak{s} \in \mathcal{S}(M)$  is a spin structure with associated characteristic sublink  $L' \subset L$ , then we denote

$$(M, \mathfrak{s}) = K_1^{(f_1; t_1)} \cup \dots \cup K_N^{(f_N; t_N)}$$

where each  $t_i \in \mathbb{Z}/2\mathbb{Z} = \{1, -1\}$  satisfies  $t_i = -1$  if and only if  $K_i \in L'$ .

From [GS99], when sliding  $K_i$  over  $K_j$ ,  $(f_i; t_i) \mapsto (f_i + f_j \pm 2\ell k(K_i, K_j); t_i)$  and  $(f_j; t_j) \mapsto (f_j; t_j)$ . Furthermore, blowing-up corresponds to the addition of  $(\pm 1; -1)$ -decorated unknot. From these two observations, we immediately conclude the following lemma.

**Lemma 3.13.** *Suppose that a band of  $k$  strands has  $r$  strands contained in the characteristic sublink of a spin structure  $\mathfrak{s}$  on  $M$  and the remaining  $k - r$  strands not in the characteristic sublink, then adding  $-s_i$ -full twists to the band, through the introduction of a canceling pair, effects the characteristic sublink as in Figure 20 with no change to the original characteristic*

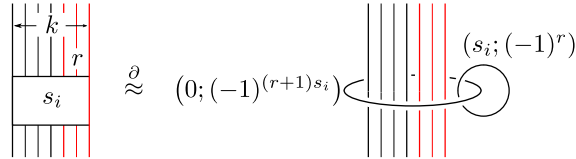


FIGURE 20. Tracing characteristic sublinks when introducing a canceling pair.

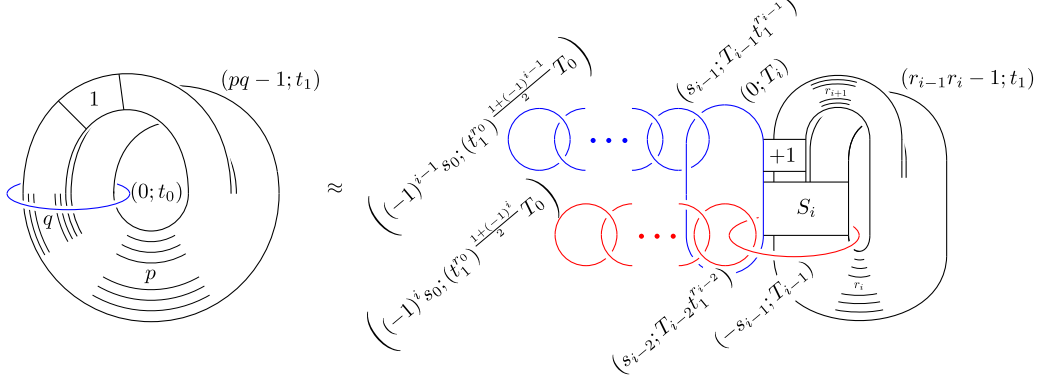
sublink and with framings within the band changing in the obvious way.

Thus, we can refine Proposition 3.1 to carry a fixed spin structure on  $\partial B_{p,q}$  to each  $\partial B_{p,q}^i$ .

**Lemma 3.14.** *Let  $\mathfrak{s} \in \mathcal{S}(\partial B_{p,q})$  be specified by the pair  $(t_0, t_1) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , then  $\mathfrak{s}$  corresponds to the spin structure on  $\partial B_{p,q}^i$  in Figure 21 where  $T_0 = t_0$  and for  $1 \leq i \leq \ell + 1$ ,  $T_i = (-1)^{1 + \det A_{i-1}} (-t_0)^{\rho_{\ell+1-i}} (t_1)^{p \det A_{i-1} + i r_i}$  such that  $A_i$  and  $\rho_{\ell+1-i}$  are as defined in Lemma 5.1.*

*Proof.* Starting with  $(t_0, t_1)$  on  $\partial B_{p,q}$ , Lemma 3.13 combined with Proposition 3.1 gives that the  $T_j$ 's in Figure 21 are defined recursively by  $T_{-1} \doteq 0$ ,  $T_0 \doteq t_0$ , and  $T_j = (-T_{j-1} t_1^{r_{j-1}})^{s_{j-1}} T_{j-2}$ . To see that the closed form for  $T_j$  is as claimed, note that we can assume  $T_j = (-1)^{a_j} (t_0)^{b_j} (t_1)^{c_j}$  for sequences  $\{a_j\}, \{b_j\}, \{c_j\} \subset \mathbb{Z}$  which only need to be determined to their respective parities. Then, the recursion on  $T_j$  descends to

$$\begin{aligned} a_{-1} &\doteq 0 & b_{-1} &\doteq 0 & c_{-1} &\doteq 0 \\ a_0 &\doteq 0 & b_0 &\doteq 1 & c_0 &\doteq 0 \\ a_j &= s_{j-1}(a_{j-1} + 1) + a_{j-2} & b_j &= b_{j-1}b_{j-1} + b_{j-2} & c_j &= s_{j-1}(c_{j-1} + r_{j-1}) + c_{j-2}. \end{aligned}$$

FIGURE 21. A fixed spin structure on  $\partial B_{p,q}$  and  $\partial B_{p,q}^i$ .

By noting that  $\rho_{\ell+1} = 1$ ,  $\rho_\ell = s_0$  and  $\rho_{\ell+1-j} = \rho_{\ell+1-(j-1)}s_{j-1} + \rho_{\ell+1-(j-2)}$  the result follows by induction on  $j$ .  $\square$

**Remark 3.15.** By Lemma 5.1, we have that  $\det A_\ell = \pm d$  for  $d$  defined therein. Thus,

$$T_{\ell+1} = (-1)^{1+d}(-t_0)^m(t_1)^{pd+\ell+1}.$$

If  $p \in 2\mathbb{Z}$ , then  $t_1 = -1$  for both spin structures on  $\partial B_{p,q}$  and we can further reduce  $T_{\ell+1}$  to  $(-1)^{c+\ell}t_0$  (as  $m$  is necessarily odd and the parities of  $c$  and  $d$  always oppose each other in this case). Therefore, when  $p \in 2\mathbb{Z}$ , we can measure which spin structure  $\mathfrak{s}$  gives on  $\partial B_{p,q}$  in the linear plumbing of Figure 11 by noting that the  $-r_\ell$ -framed unknot will be in the characteristic sublink associated to  $\mathfrak{s}$  if and only if  $(-1)^{c+\ell}t_0 = -1$ . Of course, we can also measure this by looking at the  $-s_0$ -framed unlink. However, to see which spin structure is induced on  $\partial A_{m,n}$ , it is convenient to look at  $-r_\ell$ . To that end, we have

**Proposition 3.16.** *Let  $\mathfrak{s}$  be the spin structure on  $\partial B_{p,q}$  specified by  $(t_0, t_1)$ , then  $f_*(\mathfrak{s})$  is the spin structure on  $\partial A_{m,n}$  specified by*

$$(v_0, v_1) = \left( \frac{(-1)^{c+\ell}t_0 + t_1 + (-1)^{c+\ell+1}t_0t_1 + 1}{2}, t_1 \right)$$

where the pair  $(v_0, v_1) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  is analogously defined for  $\partial A_{m,n}$  as the pair  $(t_0, t_1)$  is in Figure 21 for  $\partial B_{p,q}$ .

Using Proposition 3.16, we can deduce Corollary 1.3 by carving. Carving is a powerful tool for understanding handlebody decompositions (see, for instance, [Akb77] and [Akb14]). The fact that carving  $f(\mu_1)$  gives  $S^1 \times B^3$  is enough to extend  $f$  to a diffeomorphism between  $B_{p,q}$  and  $A_{m,n}$ :

*Proof of Corollary 1.3.* By Theorem 1.2, there exists  $f : \partial B_{p,q} \rightarrow \partial A_{m,n}$  satisfying that  $f$  carries the belt sphere,  $\mu_1$ , of the single 2-handle in  $B_{p,q}$  to an unknot in  $\partial A_{m,n}$ . Remarks 3.4 and 3.8 show that the 0-framing on  $\mu_1$  determined by the cocore of the 2-handle is preserved as well. Therefore,  $f$  can be defined across the cocore of the 2-handle in  $B_{p,q}$ . Thus, we can view  $f$  as giving a diffeomorphism,  $f_0$ , between the result of 0-surgery on  $\mu_1 \subset \partial B_{p,q}$  to that of  $f(\mu_1) = \gamma \subset \partial A_{m,n}$ . As carving both  $\mu_1$  and  $f(\mu_1)$  gives  $S^1 \times B^3$ ,  $f_0$  is a diffeomorphism of  $S^1 \times S^2$  to itself which extends uniquely over  $S^1 \times B^3$  since we can verify that  $f_0$  doesn't intertwine the spin structures of  $S^1 \times S^2$  by examining Proposition 3.16.  $\square$

#### 4. HOMOTOPY INVARIANTS OF THE INDUCED CONTACT STRUCTURES

In this section, we compare the homotopy invariants of the contact structures induced by  $\tilde{J}_{m,n}$  on  $\partial A_{m,n}$  to those induced by the Stein structures of  $B_{p,q}$ . The latter are known to induce

contact structures which are contactomorphic to the standard contact structure,  $(L(p^2, pq - 1), \tilde{\xi}_{st})$  - thus Lisca's classification result (Theorem 2.1) applies. For identifying tight contact structures on lens spaces, it is enough to know that the two contact structures in question are homotopic up to contactomorphism. Indeed, the following result of Honda's ensures this.

**Theorem 4.1** ([Hon00], Proposition 4.24). *The homotopy classes of the tight contact structures of  $L(p, q)$  are all distinct.*

Further, it is known for contact structures with  $c_1$  torsion (which is always satisfied for 3-manifolds with  $b_1 = 0$ ; e.g. lens spaces) that particular homotopy invariants completely determine their homotopy classes. In [Gom98], Gompf defines two invariants,  $d_3$  and  $\Gamma$ , and proves:

**Theorem 4.2** ([Gom98], Theorem 4.16). *If  $(M^3, \xi_i)$  for  $i = 1, 2$ , satisfies that  $c_1(\xi_1)$  is torsion and  $\Gamma(\xi_1, \mathfrak{s}) = \Gamma(\xi_2, \mathfrak{s})$  for some spin structure  $\mathfrak{s}$ , then  $\xi_1$  is homotopic to  $\xi_2$  if and only if their  $d_3$  invariants coincide.*

We recall the definitions of  $d_3$  and  $\Gamma$ . For the three-dimensional invariant,  $d_3$ , we use the normalized definition found in [OS04] - but note that it is equivalent to the definition of  $\theta$  in [Gom98] which relies on the fact that each contact 3-manifold can be realized as the boundary of an almost complex 4-manifold as well as the fact that for  $(X^4, J)$ , a closed almost complex 4-manifold, the quantity  $c_1^2(X, J) - 3\sigma(X) - 2\chi(X) = 0$  where  $\sigma(X)$  and  $\chi(X)$  are the signature and Euler characteristic of  $X$  respectively.

**Definition 4.3** ([Gom98], Definition 4.2). For a contact 3-manifold  $(M, \xi)$  with  $c_1(\xi)$  torsion, the three-dimensional invariant

$$d_3(\xi) = \frac{1}{4} (c_1^2(X, J) - 3\sigma(X) - 2\chi(X)) \in \mathbb{Q}$$

for any almost complex 4-manifold  $(X, J)$  with  $\partial X = M$  satisfying  $JTM \cap TM = \xi$ .

$\Gamma$  associates to each spin structure on  $(M, \xi)$  an element of  $H_1(M; \mathbb{Z})$ . This is accomplished by noting that each spin structure on  $(M^3, \xi)$  provides a trivialization of  $TM$ , which, in turn, identifies  $\text{Spin}^c(M)$  with  $H^2(M; \mathbb{Z})$ . Then, with respect to this identification,  $\Gamma(\xi, \mathfrak{s})$  is Poincaré dual to the  $\text{spin}^c$ -structure induced by  $\xi$ . If  $(M, \xi) = \partial(X, J)$ , a Stein domain, [Gom98] provides the following characterization of  $\Gamma$  that we make extensive use of. Suppose that  $(X, J)$  is obtained by attaching 2-handles to a Legendrian link  $K_1 \cup \dots \cup K_k$  in  $\partial(S^1 \times B^3 \natural \dots \natural S^1 \times B^3)$  with Seifert framings given by  $\text{tb}(K_i) - 1$ . Let  $\tilde{X}$  be the result of surgering each one handle and let  $L_0$  be the collection of 0-framed unknots, resulting from those surgeries.

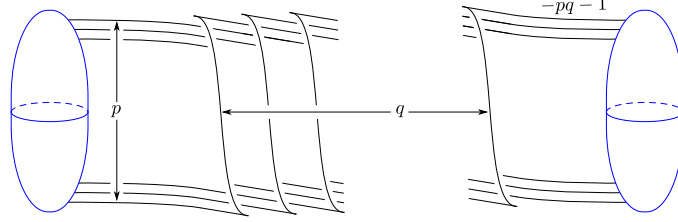
**Proposition 4.4** ([Gom98], Theorem 4.12). *Let  $(X, J)$  and  $\tilde{X}$  be defined as above. Orient  $K_1 \cup \dots \cup K_k \cup L_0$  to obtain a spanning set for  $H_2(\tilde{X}; \mathbb{Z})$ . Then  $\Gamma(\xi, \mathfrak{s}) \in H_1(\partial X; \mathbb{Z})$  is Poincaré dual to the restriction of the class  $\rho \in H^2(X; \mathbb{Z})$  whose value on each  $[K_i]$  is given by*

$$\rho([K_i]) = \frac{1}{2} (\text{rot}(K_i) + \ell k(K_i, L' + L_0)) \in \mathbb{Z}$$

where  $\text{rot}(K) = 0$  for each  $K \in L_0$  and where  $L'$  is the characteristic sublink associated to  $\mathfrak{s}$ .

**Proposition 4.5.** *For  $p > q \geq 1$  relatively prime, the contact structure induced by the Stein structure,  $J_{p,q}$ , on  $B_{p,q}$  given by Figure 22 has  $\Gamma(\xi_{J_{p,q}}, \mathfrak{s}) = \frac{pq}{2} \cdot \mu_0$  in an appropriate basis of  $H_1(L(p^2, pq - 1); \mathbb{Z})$  and for a fixed choice of  $\mathfrak{s}$  when  $p \in 2\mathbb{Z}$ .*

*Proof.* Let  $K_0$  be the boundary of the carving disk of the 1-handle in Figure 22 let  $K_1$  be the attaching circle of the single 2-handle, and let  $X_{p,q}$  be the 4-manifold obtained from Figure 22 by surgering the 1-handle (exchanging the “dot” on  $K_0$  for a 0-framed 2-handle). Then, let  $\mathfrak{s} \in \mathcal{S}(\partial B_{p,q})$  be the spin structure on  $\partial B_{p,q}$  specified by  $(t_0, t_1)$  in Figure 21. As we have to

FIGURE 22.  $(B_{p,q}, J_{p,q})$ 

slide the 2-handle under 1-handle  $q$ -times to arrive at Figure 22, we see that  $\mathfrak{s}$  corresponds to the characteristic sublink

$$L' = \frac{1 - t_0 t_1^q}{2} K_0 + \frac{1 - t_1}{2} K_1$$

in  $X_{p,q}$ . Orient the 2-handles so that  $\text{rot}(K_1) = q$  and so that  $\ell k(K_0, K_1) = p$ . In this orientation, let  $\tilde{\mu}_i$  be a right handed meridian for  $K_i$  in  $X_{p,q}$  and let  $\mu_i$  be a right handed meridian for the corresponding (oriented) knots in  $\partial B_{p,q}$  of Figure 21 so that

$$\begin{aligned} H_1(\partial X_{p,q}; \mathbb{Z}) &= \langle \tilde{\mu}_0, \tilde{\mu}_1 : p\tilde{\mu}_1 = 0, p\tilde{\mu}_0 = (pq + 1)\tilde{\mu}_1 \rangle, \\ H_1(\partial B_{p,q}; \mathbb{Z}) &= \langle \mu_0, \mu_1 : p\mu_1 = 0, p\mu_0 = (1 - pq)\mu_1 \rangle, \end{aligned}$$

where  $\tilde{\mu}_0 = \mu_0 + q\mu_1$  and  $\tilde{\mu}_1 = \mu_1$ . Then, for  $j = 0, 1$ , by Proposition 4.4, we have

$$\rho([K_j]) = \frac{1}{2} \left( \frac{1 - t_1}{2} p \right) (1 - j) + \frac{1}{2} \left( q + \frac{3 - t_0 t_1^q}{2} p - \frac{1 - t_1}{2} (pq + 1) \right) j.$$

Noting that  $\mu_1 = p\mu_0$ , we find that

$$\begin{aligned} \Gamma(\xi_{J_{p,q}}, \mathfrak{s}) &= \frac{1}{2} \left( \frac{1 - t_1}{2} p \right) \tilde{\mu}_0 + \frac{1}{2} \left( q + \frac{3 - t_0 t_1^q}{2} p - \frac{1 - t_1}{2} (pq + 1) \right) \tilde{\mu}_1 \\ &= \left( \frac{pq}{2} + \left( \frac{3 - t_0 t_1^q}{2} \right) \frac{p^2}{2} \right) \cdot \mu_0. \end{aligned}$$

Since there is no 2-torsion in  $\mathbb{Z}/p^2\mathbb{Z}$  if  $p \in 2\mathbb{Z} + 1$ ,  $p^2/2 = 0$  in that case. If  $p \in 2\mathbb{Z}$ , then we can take  $\mathfrak{s}$  corresponding to  $(t_0, t_1) = (1, -1)$ . In either case, (fixing the spin structure) we have  $\Gamma(\xi_{J_{p,q}}, \mathfrak{s}) = \frac{pq}{2} \cdot \mu_0$ .  $\square$

**Proposition 4.6.** *For  $n > m \geq 1$  relatively prime, the contact structure induced by the Stein structure  $(A_{m,n}, \tilde{J}_{m,n})$  given by Figure 2 has*

$$\Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathfrak{s})) = \frac{m+n}{2} \left( (d-c)^2 + \frac{1-t_1}{2} \left( 1 + (d-c)^2 \left( mn + \frac{1 + (-1)^{c+\ell} t_0}{2} (m+n) \right) \right) \right) \gamma_0$$

in an appropriate basis of  $H_1(\partial A_{m,n}; \mathbb{Z})$  where  $cm + dn = 1$ .

*Proof.* Let  $\tilde{X}_{m,n}$  be the 4-manifold obtained from  $A_{m,n}$  by surgering the 1-handle. Let  $f_*(\mathfrak{s}) \in \mathcal{S}(\partial A_{m,n})$  be the spin structure corresponding to the characteristic sublink  $(t_0, t_1)$  in  $\partial B_{p,q}$ . From Proposition 3.16, we have that  $f_*(\mathfrak{s}) = \left( \frac{(-1)^{c+\ell} t_0 + t_1 + (-1)^{c+\ell+1} t_0 t_1 + 1}{2}, t_1 \right)$ . Then, since we slide the 2-handle once under the 1-handle to get to Figure 2, we consider the characteristic sublink

$$L' = \frac{1 - t_1}{2} \left( \left( \frac{1 + (-1)^{c+\ell} t_0}{2} \right) K_0 + K_1 \right)$$

where  $K_0$  is the 0-framed unknot arising from the surgery and  $K_1$  is the Legendrian attaching circle of the single 2-handle. Orient  $K_0$  and  $K_1$  so that  $\text{rot}(K_1) = 1$  and so that  $\ell k(K_0, K_1) =$

$m + n$ . With respect to this orientation, let  $\gamma_i$  be a right-handed meridian for  $K_i$  (viewed in  $\partial A_{m,n}$  prior to the single handle slide). Then, by Proposition 4.4,

$$\Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathfrak{s})) = \frac{1-t_1}{2} \frac{m+n}{2} \gamma_0 + \frac{1}{2} \left( 1 + \frac{1-t_1}{2} \left( mn + \frac{1+(-1)^{c+\ell} t_0}{2} (m+n) \right) \right) \gamma_1$$

To see that  $\Gamma(\partial A_{m,n}, \mathfrak{s})$  is as claimed, note that

$$H_1(\partial A_{m,n}; \mathbb{Z}) = \langle \gamma_0, \gamma_1 : (m+n)\gamma_1 = 0, mn\gamma_1 = -(m+n)\gamma_0 \rangle.$$

Combining this with the following observation; for  $c$  and  $d$  with  $cm + dn = 1$ , we necessarily have  $c(m+n) + (d-c)n = 1$  and  $d(m+n) - (d-c)m = 1$ . Multiplying these two equations gives that

$$-(d-c)^2 \cdot mn + (cd(m+n) - c(d-c)m + d(d-c)n) \cdot (n+m) = 1.$$

Thus,

$$\begin{aligned} \gamma_1 &= \gamma_1 - (cd(m+n) - c(d-c)m + d(d-c)n) \cdot (n+m) \gamma_1 \\ &= (1 - (cd(m+n) - c(d-c)m + d(d-c)n) \cdot (n+m)) \gamma_1 \\ &= -(d-c)^2 \cdot mn \cdot \gamma_1 \\ &= (d-c)^2 \cdot (m+n) \cdot \gamma_0. \end{aligned}$$

Exchanging  $\gamma_1$  for  $(d-c)^2(m+n)\gamma_0$  in  $\Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathfrak{s}))$  gives the result.  $\square$

**Corollary 4.7.** *Suppose that  $n > m \geq 1$  and  $p - q > q \geq 1$  are each relatively prime such that  $A(p - q, q) = (m, n)$  or  $A(p - q, q) = (n, m)$ , then  $\xi_{\tilde{J}_{m,n}}$  is contactomorphic to  $\xi_{J_{p,q}}$ .*

*Proof.* We show that, after a suitable identification of  $\partial A_{m,n}$  and  $\partial B_{p,q}$ , the homotopy class of  $\xi_{\tilde{J}_{m,n}}$  corresponds with that of  $\xi_{J_{p,q}}$ . Both contact structures arise as complex tangencies of the boundaries of Stein structures on rational 4-balls. As such,

$$d_3(\xi_{\tilde{J}_{m,n}}) = d_3(\xi_{J_{p,q}}) = -\frac{1}{2}.$$

Therefore, we only need to show that by applying  $f^{-1} : \partial A_{m,n} \rightarrow \partial B_{p,q}$  of Theorem 1.2,

$$\Gamma(f_*^{-1} \xi_{\tilde{J}_{m,n}}, \mathfrak{s}) = \Gamma(\xi_{J_{p,q}}, \mathfrak{s})$$

for some spin structure  $\mathfrak{s} \in \mathcal{S}(\partial B_{p,q})$ . Now, by Proposition 4.6 along with Remark 3.10 and Lemma 5.3 we have

$$\begin{aligned} \Gamma(f_*^{-1} \xi_{\tilde{J}_{m,n}}, \mathfrak{s}) &= f_*^{-1} \Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathfrak{s})) \\ &= \frac{p}{2} \left( (d-c)^2 + \frac{1-t_1}{2} \left( 1 + (d-c)^2 \left( mn + \frac{1+(-1)^{c+\ell} t_0}{2} p \right) \right) \right) f_*^{-1}(\gamma_0) \\ &= \begin{cases} \frac{p}{2} \left( (d-c)^2 + \frac{1-t_1}{2} \left( 1 + (d-c)^2 \left( mn + \frac{1+(-1)^{c t_0}}{2} p \right) \right) \right) n \mu_0 & \text{if } \ell \in 2\mathbb{Z}, \\ \frac{p}{2} \left( (d-c)^2 + \frac{1-t_1}{2} \left( 1 + (d-c)^2 \left( mn + \frac{1+(-1)^{d t_0}}{2} p \right) \right) \right) m \mu_0 & \text{if } \ell \in 2\mathbb{Z} + 1 \end{cases} \\ &= \frac{pq}{2} \mu_0 = \Gamma(\xi_{J_{p,q}}, \mathfrak{s}) \end{aligned}$$

where the case when  $\ell \in 2\mathbb{Z} + 1$  follows from Lemma 5.3 by symmetry. It follows from Theorem 4.2 that  $\xi_{J_{p,q}}$  and  $f_*^{-1} \xi_{\tilde{J}_{m,n}}$  are in the same homotopy class and thus, by Theorem 4.1, isotopic. Therefore  $f^{-1}$  gives a contactomorphism from  $(\partial A_{m,n}, \xi_{\tilde{J}_{m,n}})$  to  $(\partial B_{p,q}, \xi_{J_{p,q}})$ .  $\square$



This completes the proof of Theorem 1.1. Although Lisca's result allows us to conclude that  $A_{m,n} \approx B_{p,q}$  whenever their boundaries coincide, it does not tell us anything about the Stein structures  $\tilde{J}_{m,n}$  versus  $J_{p,q}$ . In [LM12], the authors note that it is unknown whether or not  $B_{p,q}$  admits more than one Stein structure. Clearly, Theorem 1.1 fails to answer that question; although, it does provide another candidate for study.

## 5. THE ALGEBRAIC DETAILS

In this section we state the necessary algebra used in the proofs of Sections 3 and 4. We start by giving a definition of the function  $A$  of [Yam07] which associates the relatively prime pair  $(m, n)$  to a given relatively prime pair  $(p - q, q)$ . Rather than relying on Yamada's original definition, we provide a description of  $A$  which dovetails with the boundary diffeomorphisms of Section 3. The following lemma gives that definition and proves that it is equivalent to Yamada's original definition.

**Lemma 5.1.** *Let  $p - q > q \geq 1$  be relatively prime, and let  $\{r_i\}_{i=-1}^{\ell+1}$  and  $\{s_i\}_{i=0}^{\ell}$  be defined as in Definition 1.4. Define sequences  $\{\sigma_i\}_{i=0}^{\ell}$  and  $\{\rho_i\}_{i=-1}^{\ell+1}$  by  $\sigma_0 \doteq r_{\ell} - 1$ ,  $\sigma_i \doteq s_{\ell-i+1}$  for  $i \in \{1, \dots, \ell\}$ . Define  $\rho_i$  recursively by setting  $\rho_{\ell+1} \doteq 1$ ,  $\rho_{\ell} \doteq s_0$ , and defining*

$$\rho_i = \rho_{i+1}\sigma_{i+1} + \rho_{i+2}.$$

*Set  $m \doteq \rho_0$  and  $n \doteq \rho_{-1}$ . Then for  $m$  and  $n$  as defined, we have*

$$A(p - q, q) = \begin{cases} (m, n) & \text{if } \ell \in 2\mathbb{Z}, \\ (n, m) & \text{if } \ell \in 2\mathbb{Z} + 1. \end{cases} \quad (-1)^{\ell}(-c, d) = (|\det A_{\ell-1}| + (r_{\ell} - 1)|\det A_{\ell}|, |\det A_{\ell}|)$$

*where  $c$  and  $d$  are the unique integers, with  $0 < (-1)^{\ell+1}c, (-1)^{\ell}d < p$ , satisfying  $cm + dn = 1$ , and*

$$A_i = \begin{pmatrix} s_1 & 1 & & & \\ 1 & -s_2 & 1 & & \\ & 1 & \ddots & 1 & \\ & & 1 & (-1)^{i+1}s_i & \end{pmatrix}.$$

*Proof.* Recall the definition of  $A(p - q, q)$ , as well as the pair  $(c, d)$  in [Yam07]: Set  $(a_0, b_0) \doteq (p - q, q)$ ,  $(m_0, n_0) \doteq (1, 1)$ ,  $(c_0, d_0) = (0, 1)$ . If  $a_i > b_i$ ,

$$(a_{i+1}, b_{i+1}) \doteq (a_i - b_i, b_i), \quad (m_{i+1}, n_{i+1}) \doteq (m_i + n_i, n_i), \quad (c_{i+1}, d_{i+1}) \doteq (c_i, d_i + c_i)$$

and if  $a_i < b_i$ ,

$$(a_{i+1}, b_{i+1}) \doteq (a_i, b_i - a_i), \quad (m_{i+1}, n_{i+1}) \doteq (m_i, n_i + m_i), \quad (c_{i+1}, d_{i+1}) \doteq (c_i + d_i, d_i).$$

Then  $A(p - q, q) \doteq (m_N, n_N)$  and  $-c_N m_N + d_N n_N = 1$  for  $N$  such that  $a_N = b_N = 1$  - which exists since  $(p - q, q) = 1$ . Since  $p - q > q$ , there is a subsequence  $\{(a_{i_j}, b_{i_j})\}_{j=1}^{\ell+2} \subset \{(a_i, b_i)\}_{i=0}^N$  satisfying

$$(a_{i_j}, b_{i_j}) = \begin{cases} (r_j, r_{j-1}), & \text{if } j \in 2\mathbb{Z} + 1, \\ (r_{j-1}, r_j), & \text{if } j \in 2\mathbb{Z} \end{cases}$$

for  $j \in \{1, \dots, \ell + 1\}$ , and  $i_{\ell+2} = N$ . Furthermore, for these indicies, we have

$$(m_{i_j}, n_{i_j}) = \begin{cases} (\rho_{\ell-j+1}, \rho_{\ell-j+2}), & \text{if } j \in 2\mathbb{Z} + 1, \\ (\rho_{\ell-j+2}, \rho_{\ell-j+1}), & \text{if } j \in 2\mathbb{Z}, \end{cases}$$

Thus for  $j = \ell + 2$  we find that

$$A(p - q, q) = (m_N, n_N) = \begin{cases} (\rho_{-1}, \rho_0), & \text{if } \ell \in 2\mathbb{Z} + 1, \\ (\rho_0, \rho_{-1}), & \text{if } \ell \in 2\mathbb{Z}. \end{cases}$$

To see that this gives the claim for  $(c, d)$  as well, we note for  $j \leq \ell + 1$ , we have

$$(c_{i_j}, d_{i_j}) = \begin{cases} (|\det A_{j-2}|, |\det A_{j-1}|), & \text{if } j \in 2\mathbb{Z} + 1, \\ (|\det A_{j-1}|, |\det A_{j-2}|), & \text{if } j \in 2\mathbb{Z}. \end{cases}$$

where  $A_{-1} \doteq 0$  and  $A_0 \doteq 1$ . Now, to produce such a subsequence, take  $i_1 = s_0 - 1 > 1$  (so that  $a_i > q$  for each  $i < i_1$ ) similarly, take  $i_{k+1} = s_k + i_k$  for  $k \leq \ell$  and take  $i_{\ell+2} = i_{\ell+1} + r_\ell - 1$ . By definition,

$$(a_{i_1}, b_{i_1}) = (p - q - (s_0 - 1)q, q) = (r_1, r_0).$$

On the other hand

$$(m_{i_1}, n_{i_1}) = (1 + (s_0 - 1), 1) = (\rho_\ell, \rho_{\ell+1}), \quad (c_{i_1}, d_{i_1}) = (0, 1 + 0) = (0, 1).$$

For  $i_{k+1}$  we have (for  $k < \ell + 1$ ),

$$(a_{i_{k+1}}, b_{i_{k+1}}) = \begin{cases} (r_k, r_{k-1} - s_k r_k), & \text{if } k \in 2\mathbb{Z} + 1 \\ (r_{k-1} - s_k r_k, r_k), & \text{if } k \in 2\mathbb{Z} \end{cases} = \begin{cases} (r_k, r_{k+1}), & \text{if } k + 1 \in 2\mathbb{Z} \\ (r_{k+1}, r_k), & \text{if } k + 1 \in 2\mathbb{Z} + 1. \end{cases}$$

and  $(a_{i_{\ell+2}}, b_{i_{\ell+2}}) = (1, 1)$ . For  $k \leq \ell + 1$ ,

$$\begin{aligned} (m_{i_{k+1}}, n_{i_{k+1}}) &= \begin{cases} (\rho_{\ell-k+1}, \rho_{\ell-k+2} + s_k \rho_{\ell-k+1}), & \text{if } k \in 2\mathbb{Z} + 1 \\ (\rho_{\ell-k+2} + s_k \rho_{\ell-k+1}, \rho_{\ell-k+1}), & \text{if } k \in 2\mathbb{Z} \end{cases} \\ &= \begin{cases} (\rho_{\ell-k+1}, \rho_{\ell-k+2} + \sigma_{\ell-k+1} \rho_{\ell-k+1}), & \text{if } k \in 2\mathbb{Z} + 1 \\ (\rho_{\ell-k+2} + \sigma_{\ell-k+1} \rho_{\ell-k+1}, \rho_{\ell-k+1}), & \text{if } k \in 2\mathbb{Z} \end{cases} \\ &= \begin{cases} (\rho_{\ell-k+1}, \rho_{\ell-k}), & \text{if } k + 1 \in 2\mathbb{Z} \\ (\rho_{\ell-k}, \rho_{\ell-k+1}), & \text{if } k + 1 \in 2\mathbb{Z} + 1. \end{cases} \end{aligned}$$

Finally notice that

$$\det A_i = (-1)^{i+1} s_i \det A_{i-1} - \det A_{i-2}$$

and that the sign of  $A_i$  coincides with the sign of  $\sin(\pi i/2) + \cos(\pi i/2)$  giving that  $|\det A_i| = s_i |A_{i-1}| + |A_{i-2}|$ . Therefore,

$$\begin{aligned} (c_{i_{k+1}}, d_{i_{k+1}}) &= \begin{cases} (|\det A_{k-2}| + s_k |\det A_{k-1}|, |\det A_{k-1}|), & \text{if } k \in 2\mathbb{Z} + 1 \\ (|\det A_{k-1}|, |\det A_{k-2}| + s_k |\det A_{k-1}|), & \text{if } k \in 2\mathbb{Z} \end{cases} \\ &= \begin{cases} (|\det A_k|, |\det A_{k-1}|), & \text{if } k + 1 \in 2\mathbb{Z} \\ (|\det A_{k-1}|, |\det A_k|), & \text{if } k + 1 \in 2\mathbb{Z} + 1. \end{cases} \end{aligned}$$

When passing to  $k = \ell + 2$ , we have

$$(c_{i_{\ell+2}}, d_{i_{\ell+2}}) = \begin{cases} (|\det A_\ell|, |\det A_{\ell-1}| + (r_\ell - 1)|\det A_\ell|), & \text{if } \ell \in 2\mathbb{Z} + 1, \\ (|\det A_{\ell-1}| + (r_\ell - 1)|\det A_\ell|, |\det A_{\ell-1}|), & \text{if } \ell \in 2\mathbb{Z}. \end{cases}$$

Giving that  $(-1)^{\ell+1} (|\det A_{\ell-1}| + (r_\ell - 1)|\det A_\ell|) m + (-1)^\ell |\det A_\ell| n = 1$ .  $\square$

In general,  $c$  and  $d$  satisfying  $cm + dn = 1$  are far from unique. However, specifying them as in Lemma 5.1, (which are equivalent to the coefficients  $s$  and  $t$  that Yamada defines in [Yam07]) is crucial, since, as constructed, Yamada proves:

**Lemma 5.2** ([Yam07], Lemma 2.5). *Suppose that  $A(p - q, q) = (m, n)$ . If  $c$  and  $d$  are defined as in Lemma 5.1, giving that  $cm + dn = 1$ , then  $d - c = q$ .*

Notice that if  $A(p - q, q) = (n, m)$ , then we clearly have  $c - d = q$  instead. Lemma 5.2 allows us to simplify the quantity  $f_*^{-1} \Gamma(\xi_{\tilde{J}_{m,n}}, f_*(\mathfrak{s}))$  of Proposition 4.6. We only consider the case when  $\ell \in 2\mathbb{Z}$  (giving that  $A(p - q, q) = (m, n)$ ) since the case when  $\ell \in 2\mathbb{Z} + 1$  is symmetric by exchanging  $m \leftrightarrow n$  and  $c \leftrightarrow d$ .

**Lemma 5.3.** *Suppose that  $A(p - q, q) = (m, n)$ , and that  $cm + dn = 1$  so that  $d - c = q$ , then for  $(t_0, t_1) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , we have*

$$\frac{p}{2} \left( q^2 + \frac{1 - t_1}{2} \left( 1 + q^2 \left( mn + \frac{1 + (-1)^c t_0}{2} p \right) \right) \right) n = \frac{pq}{2}$$

in  $\mathbb{Z}/p^2\mathbb{Z}$  whenever  $p \in 2\mathbb{Z} + 1$  or when  $p \in 2\mathbb{Z}$  and  $(t_0, t_1) = (1, -1)$ .

*Proof.* Recall that  $m + n = p$  and that  $qn = 1 - cp$ . Thus, in  $\mathbb{Z}/p^2\mathbb{Z}$

$$\begin{aligned} & \frac{p}{2} \left( q(1 - cp) + \frac{1 - t_1}{2} \left( n + m(1 - cp)^2 + \frac{1 + (-1)^c t_0}{2} q(1 - cp)p \right) \right) \\ &= \frac{pq}{2} + \frac{p^2}{2} \left( -cq + \frac{1 - t_1}{2} \left( 1 - 2c + pc^2 + \frac{1 + (-1)^c t_0}{2} q \right) \right) \\ &= \frac{pq}{2} + \frac{p^2}{2} \left( -cq + \frac{1 - t_1}{2} \left( 1 + \frac{1 + (-1)^c t_0}{2} q \right) \right). \end{aligned}$$

If  $p \in 2\mathbb{Z} + 1$ , then  $\mathbb{Z}/p^2\mathbb{Z}$  lacks 2-torsion so that  $p^2/2 = 0$ . Suppose that  $p \in 2\mathbb{Z}$  and that  $(t_0, t_1) = (1, -1)$ , then the above reduces to

$$\frac{pq}{2} + \frac{p^2}{2} \left( -cq + 1 + \frac{1 + (-1)^c}{2} q \right) = \frac{pq}{2}$$

since in this case,  $q \in 2\mathbb{Z} + 1$  and the quantity  $-cq + 1 + \frac{1 + (-1)^c}{2} q$  is necessarily even.  $\square$

The following result is used to independently verify that  $\partial B_{p,q} \approx L(p^2, pq - 1)$ . To that end, we inductively build the linear plumbing of Figure 11 from the middle out. Furthermore, we choose signs on the weights so that  $-s_0$  ends up on the left. Since, a fortiori, we have

$$[-s_0, s_1, \dots, \pm r_\ell, 1, \mp r_\ell, \dots, -s_1, s_0] = \frac{\det Q_{S_{\ell+1}}}{\det Q_{S_\ell^-}} = \frac{(-1)^\ell r_{-1}^2}{(-1)^\ell (1 - r_{-1} r_0)} = \frac{-p^2}{pq - 1}$$

where we use that if  $[c_1, \dots, c_n] = -p/q$  then  $-p/q = \det C_n / \det C_{n-1}$  for the matrices  $C_i$  defined in Lemma 3.9.

**Lemma 5.4.** *Define  $\{r_i\}_{i=-1}^{\ell+2}$  and  $\{s_i\}_{i=0}^{\ell+1}$  as in Definition 1.4, let  $S_i$  be the 4-manifold given by plumbing  $D^2$ -bundles over  $S^2$  according to the weighted graph in Figure 23. Let  $S_i^+$  be the*

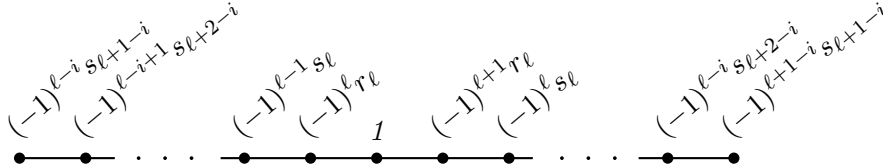


FIGURE 23. The 4-manifold  $S_i$ .

4-manifold obtained by plumbing an Euler class  $(-1)^{\ell-i-1} s_{\ell-i}$  disk bundle to the Euler class  $(-1)^{\ell-i} s_{\ell+1-i}$  disk bundle in  $S_i$ . Let  $S_i^-$  be the 4-manifold obtained by plumbing an Euler class  $(-1)^{\ell-i} s_{\ell-i}$  disk bundle to the Euler class  $(-1)^{\ell+1-i} s_{\ell+1-i}$  disk bundle in  $S_i$ . Then the

intersection forms of  $S_i$  and  $S_i^\pm$  satisfy

$$\begin{aligned}\det Q_{S_i} &= (-1)^{i+1} r_{\ell-i}^2, \\ \det Q_{S_i^+} &= (-1)^\ell \left( r_{\ell-i-1} r_{\ell-i} + (-1)^{\ell+i} \right), \\ \det Q_{S_i^-} &= (-1)^\ell \left( (-1)^{\ell+i} - r_{\ell-i-1} r_{\ell-i} \right).\end{aligned}$$

*Proof.* Induct on  $i$  by noting that

$$\begin{aligned}\det Q_{S_i^\pm} &= (-1)^{\ell-i-(1\pm 1)/2} s_{\ell-i} \det Q_{S_i} - \det Q_{S_{i-1}^\mp}, \\ \det Q_{S_{i+1}} &= (-1)^{\ell-i-1} s_{\ell-i} \det Q_{S_i^-} + (-1)^{\ell-i+1} s_{\ell-i} \det Q_{S_{i-1}^-} + \det Q_{S_{i-1}},\end{aligned}$$

as well as the fact that, by definition,  $r_k = r_{k+1} s_{k+1} + r_{k+2}$ .  $\square$

Finally, Lemma 3.9 requires that we understand certain determinants arising from the intersection form of a given linear plumbing. For the examples considered, we calculate those determinants here - they are used to express the generator,  $\gamma_0$ , of  $H_1(\partial A_{m,n})$  in terms of  $\mu_0 \in H_1(\partial B_{p,q})$ .

**Lemma 5.5.** *Let  $\{\rho_i\}_{i=-1}^{\ell+2}$  and  $\{\sigma_i\}_{i=0}^{\ell+1}$  be as defined in Definition 1.4, (associated to  $n$  and  $m$ ) then for each  $i \leq \ell + 1$  we have*

$$\det \begin{pmatrix} -\rho_\ell & 1 & & & \\ 1 & \sigma_\ell & 1 & & \\ & 1 & \ddots & 1 & \\ & & 1 & (-1)^{\ell+1-i} \sigma_{\ell+1-i} & \end{pmatrix} = - \left( \sin \left( \frac{\pi}{2} i \right) + \cos \left( \frac{\pi}{2} i \right) \right) \rho_{\ell-i}.$$

*Proof.* Induct on  $i$ , using that  $\rho_{\ell+1} = 1$  and that  $\rho_{\ell-i} = \rho_{\ell-i+1} \sigma_{\ell-i+1} + \rho_{\ell-i+2}$ .  $\square$

**Corollary 5.6.** *Let  $\gamma_0, \eta_{\pm 1}$  each be meridians indicated in Figure 24. Then, fixing orientations*

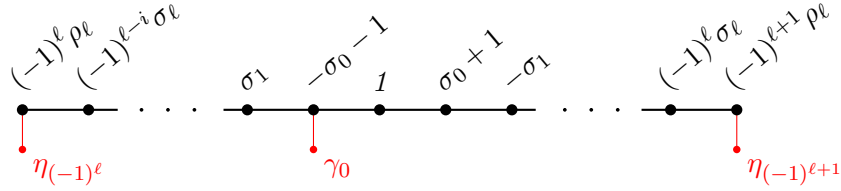


FIGURE 24. Expressing  $\gamma_0$  in terms of a “preferred” generator,  $\eta_{-1}$ , for the lens space  $\partial A_{m,n}$ .

so all linking is non-negative, we have

$$- \left( \sin \left( \frac{\pi}{2} \ell \right) + \cos \left( \frac{\pi}{2} \ell \right) \right) m \cdot \eta_{(-1)^\ell} = \gamma_0 = - \left( \sin \left( \frac{\pi}{2} \ell \right) + \cos \left( \frac{\pi}{2} \ell \right) \right) n \cdot \eta_{(-1)^{\ell+1}}.$$

*Proof.* This follows immediately from Lemma 3.9 and Lemma 5.5.  $\square$

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